Wilson loops in $\mathcal{N} = 4$ SYM and fermion droplets

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ABSTRACT: The matrix models which are conjectured to compute the circle Wilson loop and its correlator with chiral primary operators are mapped onto normal matrix models. A fermion droplet picture analogous to the well-known one for chiral primary operators is shown to emerge in the large N limit. Several examples are computed. We find an interesting selection rule for the correlator of a single trace Wilson loop with a chiral primary operator. It can be non-zero only if the chiral primary is in a representation with a single hook. We show that the expectation value of the Wilson loop in a large representation labelled by a Young diagram with a single row has a first order phase transition between a regime where it is identical to a large column representation and a regime where it is a large wrapping number single trace Wilson loop.

KEYWORDS: Matrix Models, AdS-CFT Correspondence, Supersymmetric gauge theory.

Contents

1. Introduction

The basic gauge invariant operator in Yang-Mills theory is the Wilson loop. It is used as a diagnostic of the dynamical behavior of the gauge theory. The area law of the Wilson loop as a signal of confinement is a well known example. It has also provided an interesting tool in the AdS/CFT duality where it is the Yang-Mills operator which is the most direct probe of fundamental strings [1, 2].

Recently, highly symmetric Wilson loops have attracted considerable attention due the intriguing possibility that their AdS/CFT duals are described by symmetric 3-branes and 5-branes [3]–[10]. This is in analogy with large representation chiral primary operators of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory which are interpreted as giant gravitons [11 – 13]. In the latter case, there is a beautiful picture of "bubbling geometries" which relates geometrical data of the space-time on the string theory side of AdS/CFT to the representations of SU(N) which are used to characterize chiral primary operators [14].

In that picture, the set of $\frac{1}{2}$-BPS chiral primary operators, the bosonic part of whose symmetry group is SO(4) × SO(4) × R^1, are $\mathrm{Tr}_R(z(x))$ where R is an irreducible representation of the GL(N,C) Lie algebra. An operator has U(1) R-charge J if the Young diagram

corresponding to R has J boxes. Then, there is a 1-1 correspondence between Young diagrams and Slater determinants, which are the wave-functions for quantum states of N free fermions in a harmonic potential. Due to the Pauli exclusion principle, the fermions occupy distinct states and in the appropriate large N limit, their semi-classical trajectories coalesce to form an incompressible droplet in the single-fermion phase space [15]. An analogy is then drawn between the shape of this droplet and the initial data that is necessary to specify $\frac{1}{2}$-BPS, $SO(4) \times SO(4) \times R^1$-symmetric solutions of IIB supergravity, which are interpreted as the background space-times in which the giant graviton is embedded [14].

In this paper, we shall further develop the incompressible droplet model so that it can be used to compute the circle Wilson loop, multi-wrapped and multi-trace circle loops and correlation functions between circle loops and chiral primary operators. We do this by mapping the Hermitian matrix model that is conjectured to describe the circular loop and the complex matrix model which describes the correlation function of a circular loop and a chiral primary operator onto normal matrix models. A normal matrix model is a model for complex eigenvalues. In the large N limit, the eigenvalue density forms a droplet in the complex plane. We argue that this droplet is identical to the one which characterizes chiral primary operators and which describes the states of matrix quantum mechanics [15]. Our normal matrix model integrals simply compute overlaps between wave-functions of the matrix oscillator in a coherent state basis.

One of the results that we shall find is that the operator product expansion of the fundamental representation Wilson loop for a small circle $W_\square[\text{circle}]$ contains the chiral primary operator $\text{Tr}_R z(x)$ only when R is a representation of the gauge group with a single hook, for example $R = \begin{array}{c}\square\square\square\\\square\\\square\end{array}$. Recall that, in the AdS/CFT correspondence, the Wilson loop is the source of a string world-sheet and the chiral primary operator is a graviton, or giant graviton if R is big. Our result implies that the integral of the vertex operator that couples a graviton or giant graviton to the string world-sheet vanishes unless the graviton has a particular property. At present we do not understand the origin of this selection rule, but we do observe that, as a result of it, the Wilson loop operator acts as a probe of this aspect of the structure of a graviton. A single hook giant graviton is a single spherical 3-brane extended in either the AdS_5 or S_5 direction [13].

More generally, we find that the multi-trace circle Wilson loop operator $(W_\square[\text{circle}])^K$ contains giant gravitons with up to K hooks. In the matrix model, this is an exact statement, holding for all values of N, other parameters and representations R, though the reasoning that relates it to an exact identity for the Wilson loop is most reliable in the large N limit. It is a generalization of the same known result for the matrix elements of single trace chiral primary operators and giant gravitons, $\langle \text{Tr} z^J(x) \text{Tr}_R(\bar{z}(0)) \rangle$ is non-zero only if R has a single hook [16].

In the literature, there are two different descriptions of the Wilson loop which sources a D3-brane, in ref. [3] the claim was that the large wrapping number single trace Wilson loop whereas in [6] the loop was one in a representation with a long single row Young diagram. We will show that, if λ is large enough, and in the large N limit, these two descriptions are in fact identical. The dominant contribution in an expansion of the character for the

long row representation in symmetric polynomials turns out to be the large winding single trace. We shall also observe that, if λ is lowered beyond a certain critical value ~ 5.5, there is a first order phase transition to a regime where the Wilson loop has free energy resembling the D5-brane as described in ref. [5, 6].

The loop operator of most interest in the AdS/CFT correspondence [1, 2],

$$W[C] \;=\; \frac{1}{N}\text{Tr}\mathcal{P}\exp\left[\int_C d\tau\left(iA_\mu(x(\tau))\dot{x}^\mu(\tau) + \phi(x(\tau))\cdot\hat{\theta}|\dot{x}(\tau)|\right)\right]\;, \qquad (1.1)$$

is a measure of the holonomy of a heavy W-boson which is created by a symmetry breaking condensate of the scalar fields in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory with $\hat{\theta}^i = \langle\phi^i\rangle/|\langle\phi\rangle|$. In IIB super-string theory, this operator provides a source for fundamental open strings with world-sheets bounded by the contour C, itself located on the boundary of $AdS_5 \times S^5$. In (1.1) we have written the loop appropriate to Euclidean space which will be of most interest to us in the following.

When C is a straight line or a circle, $W[C]$ is a $\frac{1}{2}$-BPS operator. It is therefore conjectured to be protected by supersymmetry. When C is a straight line, it is believed that $\langle W[\text{straight line}]\rangle = 1$. This is supported by perturbative computations up to a few orders as well as the strong coupling limit computed using AdS/CFT. This is also the case for a multiply wound loop or a product of Wilson loop operators for any array of parallel straight lines.

For the circular loop in Euclidean space, the expectation value at large N is thought to be given by the large N limit of the Gaussian matrix integral [17]

$$\langle W[\text{circle}]\rangle = \frac{\int dM\,\frac{1}{N}\text{Tr}\left(e^M\right)\,e^{-\frac{2N}{\lambda}\text{Tr}M^2}}{\int dM\,e^{-\frac{2N}{\lambda}\text{Tr}M^2}} \;\longrightarrow\; \frac{2}{\sqrt{\lambda}}I_1(\sqrt{\lambda})\text{ as }N\to\infty,\;\lambda\text{ fixed} \quad (1.2)$$

where $\lambda = g_{\text{YM}}^2 N$ is the 't Hooft coupling. When computed in an expansion at weak coupling, at small λ, this ratio is equal to the sum of ladder Feynman diagrams. It has been shown explicitly that the first few orders in diagrams which have not been included in the sum, those with internal loops, cancel identically [17 – 19]. It is conjectured, but not proven, that such corrections cancel to all orders and that the sum of ladder diagrams is the entire perturbative contribution. It is possible that instantons give a non-perturbative contribution [20, 21], so the sum of Feynman diagrams might not give the exact amplitude. However, in the infinite N, 't Hooft limit, it is plausible that instantons are suppressed and the sum of the Feynman diagrams that is summarized in (1.2) is indeed exact.

As further evidence for the consistency of this picture, when the limit of large λ is taken on the right-hand-side of eq. (1.2) it agrees with the AdS/CFT computation originally given in ref. [2], which should be valid in that limit. The circle can be obtained from a straight line by a conformal transformation. The fact that the circle and line do not have identical expectation values has been attributed to an anomaly [22]. In ref. [22], they also give a very nice argument relating, at large λ, the $\frac{1}{N^2}$ expansion of (1.2) to the moduli of Riemann surfaces in the topological expansion in string theory.

As well as the fundamental representation loop in (1.2), the sum of ladder diagrams for higher representation loops is given by simply inserting the higher representation loop

into the matrix integral. A similar equation applies to multiply wound Wilson loops or correlations of coincident loops

$$\left\langle \prod_i W_{w_i}[\text{circle}] \right\rangle = \frac{\int dM \; \prod_i \frac{1}{N}\text{Tr}\left(e^{w_i M}\right) \; e^{-\frac{2N}{\lambda}\text{Tr}M^2}}{\int dM \; e^{-\frac{2N}{\lambda}\text{Tr}M^2}} \tag{1.3}$$

Here, we emphasize, that this matrix model summarizes what is believed to be the sum of all Feynman diagrams. The complete set of $\frac{1}{2}$-BPS loops, the bosonic part of whose symmetry group is $\text{SL}(2,\mathbb{R}) \times \text{SO}(3) \times \text{SO}(5)$ is specified by considering loops in all irreducible representations of the $\text{SU}(N)$ gauge group. The Schur polynomial formula can be used to express the expectation value of an arbitrary representation in terms of a sum over products of traces of multiply wound loops, the terms in which appear in (1.3).

When probed from a distance much greater than its radius, r, a circle Wilson loop looks like a combination of local operators

$$W[\text{circle}] = \langle W[\text{circle}]\rangle \cdot \sum_\Delta \xi_\Delta \cdot (2\pi r)^\Delta \mathcal{O}_\Delta(0)$$

where the center of the loop is near $x = 0$. We are ignoring the slight anisotropy which should occur in this formula. Since $\mathcal{N} = 4$ super Yang-Mills theory is a conformal field theory, operators can be classified according to their conformal dimensions, Δ, along with other quantum numbers. The series contains both primary operators and descendants. The coefficients ξ_Δ of primary operators can be deduced from the asymptotics of the two-point function of the loop operator and the primary field,

$$\frac{\langle W[\text{circle}]\mathcal{O}_\Delta(x)\rangle}{\langle W[\text{circle}]\rangle} = \langle \mathcal{O}_\Delta(0)\mathcal{O}_\Delta(x)\rangle \xi_\Delta \cdot (2\pi r)^\Delta + \text{ higher powers of } 1/x \tag{1.4}$$

Since the loop contains descendants as well as primary operators, the correlator with the primary operator is given by the leading asymptotic behavior at large $|x|$. The descendants would appear in higher powers of $1/|x|$.

For example, consider the chiral primary operator (we denote the symmetric traceless combination of $\text{SO}(6)_R$ indices by $(i_1 \cdots i_k)$)

$$\mathcal{O}^{(i_1 i_2 \ldots i_k)}(x) = \text{Tr}\left[\phi^{(i_1)}(x)\phi^{i_2}(x)\ldots\phi^{i_k)}(x)\right].$$

This is a $\frac{1}{2}$-BPS operator which has an exact conformal dimension $\Delta = k$ protected by supersymmetry. Two- and three-point functions of chiral primary operators are independent of the Yang-Mills coupling constant. We are most interested in chiral primary operators of a particular kind: if we take $\hat{\theta} = (1, 0, \ldots)$ in (1.1) and consider

$$\mathcal{O}_J(x) = \text{Tr}(z^J(x)) \;, \quad z(x) = \phi^1(x) + i\phi^2(x) \tag{1.5}$$

which has exact conformal dimension $\Delta = J$.

The correlation function of this operator with a circle Wilson loop located at the origin has the form

$$\frac{\langle W[\text{circle}]\mathcal{O}_J(x)\rangle}{\langle W[\text{circle}]\rangle \cdot |O_J|^2} = \left(\frac{2\pi r}{4\pi^2 x^2}\right)^J \cdot \xi_J \quad \text{as } |x| \to \infty \tag{1.6}$$

The normalization of the chiral primary operator is defined by its 2-point function,

$$\langle \mathcal{O}_J(x)\mathcal{O}_{J'}(0)\rangle = |\mathcal{O}_J|^2 \delta_{JJ'} \left(\frac{1}{4\pi^2 x^2}\right)^J \tag{1.7}$$

and can be computed using the complex matrix integral [16]

$$|\mathcal{O}_J|^2 \delta_{JJ'} = \frac{\int d^2 z e^{-\frac{2N}{\lambda}\mathrm{Tr}\bar{z}z}\mathrm{Tr}\bar{z}^J \mathrm{Tr}z^{J'}}{\int d^2 z e^{-\frac{2N}{\lambda}\mathrm{Tr}\bar{z}z}}$$
$$= \delta_{JJ'}\left(\frac{\lambda}{2N}\right)^J \frac{1}{J+1}\left[\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\right]. \tag{1.8}$$

The constants ξ_J in (1.6) were computed in the planar, small λ limit using perturbation theory and in large λ limit using the AdS/CFT correspondence in ref. [23]. The sum of planar ladder diagrams contributing to (1.6) was found in ref. [24]:

$$\xi_J = \frac{1}{N}\frac{\sqrt{J\lambda}}{2^{1+\frac{J}{2}}}\frac{I_J(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \tag{1.9}$$

This was conjectured to be an exact result. The sum of planar ladders which is summarized in (1.9), when extrapolated to large λ, agrees with the AdS/CFT computation of the same coefficients ξ_J in ref. [23]. Leading corrections from other diagrams which are not ladders were also computed and shown to vanish in ref. [24], supporting the conjecture that they give the exact result. This has further been supported by comparison with large quantum number limits of strings in ref. [25].

The sum of planar ladder diagrams which contributes to (1.9) is also given by the large N limit of a matrix integral. As well as the single trace (1.5) we could consider a more general set of chiral primaries $\mathrm{Tr}_R z(x)$ where the trace is taken in an irreducible representation.[1] In general, we could consider a Wilson loop in a representation R. Its expansion in local operators could contain a chiral primary in a representation R'. The coefficient $\xi_{RR'}$ in this expansion can also be expressed as a matrix integral,

$$\xi_{RR'} = \frac{\int d^2 z \frac{1}{\dim R}\mathrm{Tr}_R\left(e^{\frac{1}{2}(z+\bar{z})}\right)\mathrm{Tr}_{R'}(z)\,e^{-\frac{2N}{\lambda}\mathrm{Tr}\bar{z}z}}{\int d^2 z \frac{1}{\dim R}\mathrm{Tr}_R\left(e^{\frac{1}{2}(z+\bar{z})}\right)\,e^{-\frac{2N}{\lambda}\mathrm{Tr}\bar{z}z}\cdot|\mathcal{O}_{R'}|^2} \tag{1.10}$$

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[1]For an irreducible representation R corresponding to a given Young Diagram with J boxes, we begin with

$$z_{i_2}^{i_1}z_{i_3}^{i_2}z_{i_4}^{i_3}\ldots z_{i_J}^{i_{J-1}}z_{i_1}^{i_J}$$

and consider linear combinations of permutations of the upper indices until they transform in the representation of the symmetric group that is associated with the Young Diagram. We then contract the indices to get a linear combination of single and multi-trace operators. For example, when $J = 2$, there are two representations

$$z_j^i z_i^j \to \frac{1}{2}\left(z_j^i z_i^j + z_j^j z_i^i\right) \to \frac{1}{2}\mathrm{Tr}z^2 + \frac{1}{2}\mathrm{Tr}z\mathrm{Tr}z = \mathrm{Tr}_{\square\square}z$$
$$z_j^i z_i^j \to \frac{1}{2}\left(z_j^i z_i^j - z_j^j z_i^i\right) \to \frac{1}{2}\mathrm{Tr}z^2 - \frac{1}{2}\mathrm{Tr}z\mathrm{Tr}z = -\mathrm{Tr}_{\square\!\square}z$$

On the string theory side, chiral primary operators correspond to supergravitons and their coupling to loop operators can be extracted from asymptotics of the connected correlator of two far separated loop operators, which are sources for fundamental string worldsheets sitting at the boundary of $AdS_5 \times S^5$. For small representations, they should give information about the coupling of gravitons to the fundamental string world-sheets that are sourced by the loop operators. For bigger representations they would yield information about the gravitational interactions of the more exotic objects, D3-branes and D5-branes, that are sourced by the loops. When R and R' are both large, (1.10) gives information about the coupling of giant strings to giant gravitons.

In this Paper, we will take a closer look at some of the properties of the matrix integrals (1.2) and (1.10). In order to study them at large N, we will rewrite them as models for complex eigenvalues, which are called Normal matrix models.

1.1 Eigenvalue models

In some cases, following the technique of Ginibre [26], an integral over complex matrices can be written as an integral over complex eigenvalues where the matrices are diagonal and, with the suitable measure, we are left with integrating over the diagonal elements only. The most straightforward case is when the objects inside traces in the integrand are holomorphic in the complex matrix. An example is the integral which has to be done to find the normalization of the chiral primary operators

$$\langle \mathcal{O}_R \mathcal{O}_{R'} \rangle = \frac{\int d^2 z e^{-\frac{2N}{\lambda} \text{Tr} \bar{z} z} \text{Tr}_R \bar{z} \text{Tr}_{R'} z}{\int d^2 z e^{-\frac{2N}{\lambda} \text{Tr} \bar{z} z}} \tag{1.11}$$

The matrices can be made upper-triangular by a unitary transformation. The upper-triangular property ($z_{ij} = 0$ if $i > j$) is preserved by products of z's and the trace of a product of the z's depends only on diagonal matrix elements, $\text{Tr}(z^J) = \sum_i z_{ii}^J$. In the following, we will use one index to denote diagonal components, $z_i \equiv z_{ii}$. The conjugate matrix, $\bar{z}$, is lower-triangular and similar arguments apply.

The matrix integral can thus be written as an integral over diagonal components times the appropriate Jacobian times an integral over the components in the upper triangle and an integration over unitary matrices. The integration of the latter two decouples and is common between the numerator and denominator in (1.11). They cancel, leaving the integral over diagonal components with the Jacobian. The resulting measure is

$$\prod_{i=1}^{N} d^2 z_i \; \Delta(z) \Delta(\bar{z}) \tag{1.12}$$

with the vandermonde determinant

$$\Delta(z) = \prod_{i<j} (z_i - z_j) = \det z_i^{j-1} = \det z_i^{N-j}$$

When the matrix is triangular, the trace of a matrix in an irreducible representation R, $\text{Tr}_R z$ is a function only of its diagonal components and is given by the Schur polynomial

$$\text{Tr}_R z = \frac{\det(z_i^{h_j + N - j})}{\det(z_i^{N-j})} = \frac{\det(z_i^{h_j + N - j})}{\Delta(z)} \tag{1.13}$$

Here $h_1 \geq h_2 \cdots \geq h_N$ are the lengths of the rows of the Young diagram associated with the representation R.

It is now easy to see that that (1.11) can be written as an integral for diagonal elements of matrices, and then performed explicitly,

$$
\begin{aligned}
\langle \mathcal{O}_R \mathcal{O}_{R'} \rangle &= \frac{\int \prod_{i=1}^N d^2 z_i e^{-\frac{2N}{\lambda}|z_i|^2} \, |\Delta(z)|^2 \, \frac{\det(\bar{z}_i^{h_j+N-j})}{\Delta(\bar{z})} \, \frac{\det(z_i^{h'_j+N-j})}{\Delta(z)}}{\int \prod_{i=1}^N d^2 z_i e^{-\frac{2N}{\lambda}|z_i|^2} \, |\Delta(z)|^2} \\
&= \frac{\sum_{\sigma,\tau \in S_N} (-1)^{\deg\sigma + \deg\tau} \int \prod_{i=1}^N d^2 z_i e^{-\frac{2N}{\lambda}|z_i|^2} \, (\bar{z}_{\sigma(i)}^{h_\sigma(i)+N-\sigma(i)})(z_i^{h'_{\tau(i)}+N-\tau(i)})}{\sum_{\sigma,\tau \in S_N} (-1)^{\deg\sigma + \deg\tau} \int \prod_{i=1}^N d^2 z_i e^{-\frac{2N}{\lambda}|z_i|^2} \, (\bar{z}_i^{N-\sigma(i)})(z_i^{N-\tau(i)})} \\
&= \delta_{RR'} \prod_{j=1}^N \left(\frac{\lambda}{2N} \right)^{h_j} \frac{(h_j+N-j)!}{(N-j)!}
\end{aligned}
\tag{1.14}
$$

where S_N is the set of all permutations of the integers $(1,\ldots,N)$ and we have decomposed the determinants into sums over permutations using the formula

$$
\det M = \sum_{\sigma \in S_N} (-1)^{\deg\sigma} \prod_{i=1}^N M_i^{\sigma(i)}
$$

In the large N limit, if we assume that $\frac{1}{N} \sum h_i \to 0$, the result of (1.14) reduces to

$$
|O_R|^2 \to \left(\frac{\lambda}{2} \right)^{\sum_i h_i} \quad \text{as} \quad N \to \infty
$$

In the remainder of this paper, we are going to study matrix integrals such as (1.10) where making the matrices upper-triangular does not decouple the eigenvalues. What we will find is that, in many cases, we can nevertheless map it onto an eigenvalue integral over complex eigenvalues, that is, a Normal Matrix model.

2. Normal matrix model

A normal matrix model [27, 28] is a complex matrix model where the real and imaginary parts of the matrices are constrained to commute with each other. They can be simultaneously diagonalized by conjugation with a unitary matrix, $z = U \cdot \text{diag}(z_1, \ldots, z_N) U^\dagger$ and $\bar{z} = U \text{diag}(\bar{z}_1, \ldots, \bar{z}_N) U^\dagger$ where $UU^\dagger = \mathcal{I}$. Then, the integration of any function of the matrices which is invariant under $(z.\bar{z}) \to (UzU^\dagger, U\bar{z}U^\dagger)$ is taken as an integral over the diagonal elements z_i with measure given in (1.12) and the matrix z_{ij} replaced by the diagonal one $\text{diag}(z_1, \ldots z_N)$. Normal matrix models have recently found a number of interesting applications [29]–[32]. They were discussed in the context of non-critical string theory in refs. [33]–[35].

The normal matrix model and the complex matrix model are equivalent if, as is the case in (1.11) solved above, aside from the Gaussian term, the quantities in the integrand are holomorphic in the matrices. However, in the case of the Wilson loop (1.2) or the

correlator (1.10), the integrand is not holomorphic and it is less obvious that it can be written as an eigenvalue model.

Nevertheless, there are circumstances where the normal matrix model is equivalent to the complex matrix model. In the remainder of this Paper, we will examine some of these examples. The interesting feature of a normal matrix model is that, in the large N limit, the distribution of eigenvalues has support on a two-dimensional subset of the complex plane. This will lead to a droplet model for the eigenvalues.

2.1 Fundamental representation Wilson loop

As an example, consider the expectation value of the Wilson loop in the fundamental representation which is given by the Hermitian matrix integral (1.2) of the Wilson loop in this representation,

$$\langle W_{\square} \rangle = \frac{\int [dM]\, e^{-\frac{2N}{\lambda}\mathrm{Tr}M^2}\, \frac{1}{N}\mathrm{Tr}\left(e^M\right)}{\int [dM]\, e^{-\frac{2N}{\lambda}\mathrm{Tr}M^2}} \tag{2.1}$$

We can easily write this as a complex matrix model. The approach (which we shall not follow) would be to make this into a complex matrix model by introducing the imaginary part as another Gaussian matrix M_I, so that, $z = M + iM_I$ and writing

$$\langle W_{\square} \rangle = \frac{\int [d^2z]e^{-\frac{2N}{\lambda}\mathrm{Tr}\bar{z}z}\frac{1}{N}\mathrm{Tr}\left(e^{\frac{1}{2}(z+\bar{z})}\right)}{\int [d^2z]e^{-\frac{2N}{\lambda}\mathrm{Tr}\bar{z}z}} \tag{2.2}$$

It is not obvious how to now write (2.2) as an integral over eigenvalues. Backtracking, we recall that we could also have written (2.1) as an integral over the eigenvalues of Hermitian matrices,

$$\langle W_{\square} \rangle = \frac{\int \prod_i dm_i e^{-\frac{2N}{\lambda}m_i^2}\Delta^2(m)\ (e^{m_1})}{\int \prod_i dm_i e^{-\frac{2N}{\lambda}m_i^2}\Delta^2(m)} \tag{2.3}$$

To proceed, we realize that the integral that we must do in (2.3) is formally the expectation value of the operator e^{m_1} in the quantum state given by the wave-function $\psi(m_i) = \Delta(m)e^{-\frac{N}{\lambda}\sum_i m_i^2}$. This is just the ground state of a system of N fermions moving in a Harmonic potential well. (To see this, we must recall some properties of determinants of polynomials. This point is discussed in the appendix at the end of this section. See eq. (A.3).). There are other ways to present this integral. One of them is uses a holomorphic polarization of the oscillator phase space. In that presentation, the ground state wave-function of an N-fermion system is $\psi(z) = \Delta(z)e^{-\frac{N}{\lambda}\sum_i \bar{z}_i z_i}$ where z is a complex variable and the expectation value is

$$\langle W_{\square} \rangle = e^{-\lambda/8N} \frac{\int \prod_i d^2z_i e^{-\frac{2N}{\lambda}\bar{z}_i z_i}|\Delta(z)|^2\ \frac{1}{N}\mathrm{Tr}\left(e^{\sqrt{\frac{1}{2}}(z+\bar{z})}\right)}{\int \prod_i d^2z_i e^{-\frac{2N}{\lambda}\bar{z}_i z_i}|\Delta(z)|^2} \tag{2.4}$$

A detailed derivation of a general integral formula of this kind, eq. (A.1) is given in appendix A.

This is now a normal matrix model, written as an integral over complex eigenvalues. Note two differences between the normal model (2.4) and the complex model (2.2): the factor $e^{-\lambda/8N}$ in front of (2.4) and the factor of $\frac{1}{\sqrt{2}}$, rather than $\frac{1}{2}$ in front of $(z + \bar{z})$.

In summary, we have found three identical presentations of the same matrix model:

$$\langle W_{\square} \rangle = \frac{1}{Z_H} \int [dM] e^{-\frac{2N}{\lambda} \text{Tr} M^2} \frac{1}{N} \text{Tr} \, e^M \quad \text{(hermitian model)}$$

$$= \frac{1}{Z_C} \int [d^2 z] \, e^{-\frac{2N}{\lambda} \text{Tr} z\bar{z}} \frac{1}{N} \text{Tr} \, e^{\frac{1}{2}(z + \bar{z})} \quad \text{(complex model)}$$

$$= \frac{1}{Z_N} e^{-\frac{\lambda}{8N}} \int_{[z,\bar{z}]=0} [d^2 z] \, e^{-\frac{2N}{\lambda} \text{Tr} z\bar{z}} \frac{1}{N} \text{Tr} \, e^{\frac{1}{\sqrt{2}}(z + \bar{z})} \quad \text{(normal model)} \qquad (2.5)$$

where we have abbreviated the denominators as the Gaussian partition functions Z_H, Z_C, Z_N for Hermitian, complex and normal matrix models, respectively. We can easily check that the order λ term agrees in all three computations in (2.5):

$$\langle W_{\square} \rangle =$$

$$= 1 + \frac{1}{2N} \langle \text{Tr} \, M^2 \rangle_{\text{hermitian}} + \cdots = 1 + \frac{1}{2N} \frac{\lambda}{2N} \frac{N^2}{2} + \cdots = 1 + \frac{\lambda}{8} + \cdots,$$

$$= 1 + \frac{1}{4N} \langle \text{Tr} Z\overline{Z} \rangle_{\text{complex}} + \cdots = 1 + \frac{1}{4N} \frac{\lambda}{2N} \frac{2N^2}{2} + \cdots = 1 + \frac{\lambda}{8} + \cdots, \qquad (2.6)$$

$$= 1 + \frac{1}{2N} \langle \text{Tr} Z\overline{Z} \rangle_{\text{normal}} - \frac{\lambda}{8N} + \cdots = 1 + \frac{1}{2N} \frac{\lambda}{2N} \frac{N^2 + N}{2} - \frac{\lambda}{8N} + \cdots = 1 + \frac{\lambda}{8} + \cdots.$$

We also emphasize that the three expressions in (2.5) are exactly equal for all values of λ and N.

2.2 Fermion droplet model

Let us now consider the multiply wound circular Wilson loop with k windings. The discussion of the previous section leads to the Normal matrix integral

$$\langle W_{\square}^{(k)} \rangle = \frac{1}{Z_N} e^{-\frac{k^2 \lambda}{8N}} \int_{[z,\bar{z}]=0} [d^2 z] e^{-\frac{2N}{\lambda} \text{Tr} \bar{z}z} \frac{1}{N} \text{Tr} e^{\frac{k}{\sqrt{2}}(z + \bar{z})}$$

This can be written as an integral over diagonal matrices with

$$\langle W_{\square}^{(k)} \rangle = \frac{1}{Z_N} e^{-\frac{k^2 \lambda}{8N}} \int \prod_i d^2 z_i \exp \left(-\frac{2N}{\lambda} \sum_{i=1}^{N} \bar{z}_i z_i + \sum_{i \neq j} \ln |z_i - z_j|^2 + \frac{k}{\sqrt{2}} (z_1 + \bar{z}_1) \right)$$

In the large N limit, the integration over the eigenvalues $z_2, \ldots, z_N$ is dominated by a saddle point where the eigenvalues satisfy the saddle point equation

$$\frac{2}{\lambda} z_i = \frac{1}{N} \sum_{j \neq i} \frac{1}{\bar{z}_i - \bar{z}_j} + \frac{k}{\sqrt{2}N} \delta_{i1}$$

If we introduce the normalized eigenvalue density

$$\rho(z) = \frac{1}{N-1} \sum_{i=2}^{N} \delta^2(z - z_i) \quad , \quad \int d^2 z \rho(z) = 1$$

the saddle point equation

$$\frac{2}{\lambda}z \;=\; \frac{N-1}{N}\int d^2z'\frac{\rho(z')}{\bar{z}-\bar{z}'} + \frac{1}{N}\frac{1}{\bar{z}-\bar{z}_1} \quad z \in \text{support of } \rho \qquad (2.7)$$

There are two different regimes that we can analyze:

1. *Small winding loop*: $\lim_{N\to\infty}\frac{k}{N} = 0$

   In this regime, the last term in the saddle point equation (2.7) is irrelevant and the equation is solved by the constant density droplet

$$\rho(z) = \begin{cases} \frac{2}{\pi\lambda} & |z| < \sqrt{\frac{\lambda}{2}} \\ 0 & |z| > \sqrt{\frac{\lambda}{2}} \end{cases} \qquad (2.8)$$

   There are two ways to compute the expectation value of the multiply wound Wilson loop. We can first simply use the formula for evaluating the expectation value of the trace of matrices once the density is known:

$$\langle W_{\square}^{(k)}\rangle = \int d^2z\,\rho(z)e^{\frac{1}{\sqrt{2}}k(z+\bar{z})} = \frac{2}{\pi\lambda}\int_0^{\sqrt{\frac{\lambda}{2}}} r dr \int_0^{2\pi} d\theta \; e^{\sqrt{2}kr\cos\theta}$$

$$= \frac{1}{\pi}\int_0^1 t dt \int_0^{2\pi} d\theta e^{k\sqrt{\lambda}t\cos\theta} = \int_0^1 2t dt\, I_0(k\sqrt{\lambda}t) = \frac{2}{\sqrt{\lambda}k}I_1(k\sqrt{\lambda}) \quad (2.9)$$

   This is valid when $k \ll N$ and it agrees with the Hermitian Gaussian matrix model (1.2).[2]

   Alternatively, we can use the large N limit to integrate out the variables $z_2,\ldots,z_N$ to get an effective theory for z_1 interacting with the droplet. The result is an integral for z_1 with effective action

$$S_{\text{eff}} = \frac{2N}{\lambda}\bar{z}_1 z_1 - (N-1)\int d^2z \ln|z_1 - z|^2 \rho(z) + \frac{k}{\sqrt{2}}(z_1 + \bar{z}_1)$$

   This action has an electrostatic interpretation. The fist term is the potential due to a constant charge density on a large disc and it tends to attract the oppositely charged particle with position z_1 to the center of the disc. The second is the repulsive two-dimensional coulomb interaction with the charge distribution, which, using (2.8), is also a charged disc centered on the origin. We expect some cancellation of the attraction in the first term and the repulsion in the second term, particularly in the interior of the discs. The third term is the contribution of an external electric field which pulls the z_1 to the right in the complex plane. Using the constant density droplet,

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[2]The expectation value of the single trace Wilson loop is known explicitly at finite N [3]. A useful formula is a contour integral representation [36]

$$\langle W_{\square}^{(k)}\rangle = \frac{2}{k\sqrt{\lambda}}\oint \frac{d\xi}{2\pi i}e^{\frac{k\sqrt{\lambda}}{2}\xi}\left(\frac{1+\frac{k\sqrt{\lambda}}{4N\xi}}{1-\frac{k\sqrt{\lambda}}{4N\xi}}\right)^N \qquad (2.10)$$

$$\int d^2 w \rho(w) \log |w-z|^2 \;=\; \begin{cases} \frac{2}{\lambda}|z|^2 + \ln\frac{\lambda}{2} - 1 & |z| < \sqrt{\frac{\lambda}{2}} \\[2mm] \ln|z|^2 & |z| > \sqrt{\frac{\lambda}{2}} \end{cases} \qquad (2.11)$$

we arrive at

$$S_{\text{eff}} = \begin{cases} \frac{2}{\lambda}\bar{z}_1 z_1 - (N-1)\left(\ln\frac{\lambda}{2} - 1\right) + \frac{k}{\sqrt{2}}(z_1 + \bar{z}_1) + \mathcal{O}(N^0) & |z_1|^2 < \frac{\lambda}{2} \\[2mm] \frac{2N}{\lambda}\bar{z}_1 z_1 - (N-1)\ln|z_1|^2 + \frac{k}{\sqrt{2}}(z_1 + \bar{z}_1) + \mathcal{O}(N^0) & |z_1|^2 > \frac{\lambda}{2} \end{cases} \qquad (2.12)$$

Since we only found the eigenvalue density to leading order in large N, only the order N terms in this effective action should be trusted. In the region $|z_1|^2 > \frac{\lambda}{2}$, the effective action has a minimum at its minimal value, $|z_1|^2 = \frac{\lambda}{2}$, and a steep slope, of order N as $|z_1|$ increases. We conclude that, in the large N limit, the eigenvalue is confined to the disc $|z_1| < \frac{\lambda}{2}$. This is just the same disc where the eigenvalue distribution (2.8) has support in the infinite N limit. Inside the disc, the terms of order N in the energy are constant. To that order, the potential there is flat and, to understand the behavior of the particle, it is necessary to understand the terms of order one. However, to correctly evaluate the order one terms in the action (some of which have been written in (2.12)) we would need to properly evaluate the $1/N$ corrections to the density. If we did so, we would find that the terms of order one are also constant and the potential inside the disc is actually flat. The average of the position of the charge at z_1 over the degenerate saddle point, which is the disc, and of course gives precisely the same result as (2.9).

2. *Large winding loop*: $\lim_{N\to\infty}\frac{k}{N} = \xi$, with ξ a constant.

   In ref. [3], it is argued that the loop $W_\square^{(k)}$ with large winding k is dual to a D3-brane whose world-volume is $AdS_2 \times S^2$ embedded in AdS_5 and with k units of electric flux on the D3-brane. It is argued in [6] that the $AdS_2 \times S^2$ D3-brane corresponds to a Wilson loop in the symmetric representation $W_{\square\square\cdots\square}$.

   In the following, we will analyze the large winding number limit of a single trace Wilson loop. In particular, the relation between D3-brane and large winding loop is checked in the limit

$$k, N \to \infty \quad \text{with} \quad \kappa = \frac{k\sqrt{\lambda}}{4N} \quad \text{fixed} \qquad (2.13)$$

   In this case, the saddle-point equation (2.7) still holds and again the solution at leading order in large N is the disc-shaped droplet (2.8). In the effective action (2.12), there are now terms of order N in the action inside the disc as well as outside and it is these terms which must be taken into account to get the leading order in large N. The integral over z_1 has the form

$$\langle W_\square^{(k)}\rangle \sim e^{-N\lambda\xi^2/8}\left(\int_{|z_1|^2 < \frac{\lambda}{2}} d^2 z_1 e^{N\ln(\lambda/2) - N + N\frac{\xi}{\sqrt{2}}(z_1 + \bar{z}_1)}\right.$$

$$\left. + \int_{|z_1|^2 > \frac{\lambda}{2}} d^2 z_1 e^{-\frac{2N}{\lambda}|z_1|^2 + N\ln|z_1|^2 + N\frac{\xi}{\sqrt{2}}(z_1 + \bar{z}_1)}\right) \qquad (2.14)$$

In the large N limit, the classical position of the charge z_1 is outside of the disc. The second term in (2.14) dominates and can be evaluated by saddle-point integration. The saddle point is

$$z_1^* = \sqrt{\frac{\lambda}{2}} \left(\kappa + \sqrt{1 + \kappa^2} \right) = \sqrt{\frac{\lambda}{2}} \, e^{2 \sinh^{-1} \kappa} \tag{2.15}$$

The integral is

$$\langle W_{\square}^{(k)} \rangle \;\sim\; e^{2N \left(\kappa \sqrt{\kappa^2 + 1} + \sinh^{-1} \kappa \right)} \;,\quad \kappa = \frac{k \sqrt{\lambda}}{4N} \tag{2.16}$$

This expression agrees, as it should, with one derived in ref. [3] using the Hermitian matrix model. It was also shown in ref. [3] that this agrees with the Born-Infeld action of the D3-brane. It was further argued there that, for a large winding number Wilson loop, the fundamental string world-sheet blows up into a D3-brane.

In our matrix model picture, the D3-brane corresponds to an isolated eigenvalue residing outside of the droplet. This has a strong analogy with the bubbling geometry where a giant graviton, which is also a D3-brane is also described in this way [14].

3. *intermediate case*: $k \sim N^{\alpha}$ with $0 < \alpha < 1$

In this case, the electric field due to the Wilson loop term in the action is not strong enough to pull the charge out of the disc, but in the limit $N \to \infty$ it is pulled to the edge where $z_1 = \bar{z}_1 = \sqrt{\frac{\lambda}{2}}$.

The essential difference that we have found between the small and large winding number Wilson loop is that the eigenvalue which interacts with the loop resides inside the droplet for small winding number and outside of the droplet for large winding number.

3. Antisymmetric representations

We showed in the previous section that, when the winding number k of the loop was small, it was described by an eigenvalue moving in a flat potential in the interior of the droplet. When k was larger, of order N, the particle moved outside of the droplet and is interpreted as a D3-brane. In both cases, the expectation value of the Wilson loop was given by a certain electrostatic interaction energy of the particle with the rest of the droplet.

In the dual string theory, the small winding loop is well described by the boundary of fundamental string in $AdS_5 \times S^5$. We will now consider the situation when large number of such small winding loops are put on top of each other. For definiteness, let us consider the loop $\langle W_{A_K} \rangle$ in the Kth anti-symmetric representation. In refs. [5, 6], it is argued that W_{A_K} is dual to a D5-brane of the shape $AdS_2 \times S^4$ with the K unit of electric flux on it.

The Kth antisymmetric representation is the Young diagram with one column of K boxes,

$$\mathcal{A}_K \;=\; \left. \begin{array}{c} \square \\ \square \\ \square \\ \square \\ \square \end{array} \right\} \; K \text{ boxes}$$

and we consider the expectation value of the Wilson loop in this representation,

$$\langle W_{\mathcal{A}_K} \rangle = \frac{\int [dM] e^{-\frac{2N}{\lambda}\mathrm{Tr}M^2} \frac{1}{\dim \mathcal{A}_K} \mathrm{Tr}_{\mathcal{A}_K}\left(e^M\right)}{\int [dM] e^{-\frac{2N}{\lambda}\mathrm{Tr}M^2}} \tag{3.1}$$

where $\dim \mathcal{A}_K = N!/K!(N-K)!$.[3] This can be written as an integral over the eigenvalues of Hermitian matrices,

$$\langle W_{\mathcal{A}_K} \rangle = (-1)^{K+1} \frac{\int \prod_i dm_i e^{-\frac{2N}{\lambda}m_i^2} \Delta^2(m) \left(e^{\sum_{i=1}^K m_i}\right)}{\int \prod_i dm_i e^{-\frac{2N}{\lambda}m_i^2} \Delta^2(m)} \tag{3.2}$$

Where we have used the Schur polynomial formula which gives us

$$\mathrm{Tr}_{\mathcal{A}_K}\left(\mathrm{diag}M\right) = (-1)^{K+1} \sum_{i_1 < i_2 < \ldots < i_K} M_{i_1} M_{i_2} \ldots M_{i_K}$$

(See eq. (1.14) for the example of $K = 2$.) Note that, in eq. (3.3), the combinatorial factor that we would find from choosing the first K eigenvalues cancels with the dimension of the representation.

Once the Hermitian matrix integral is written in the form (3.2), we can apply the integral formula (A.1) to write the rewrite it as a normal matrix model:[4]

$$\langle W_{\mathcal{A}_K} \rangle = (-1)^{K+1} e^{-K\lambda/8N} \frac{\int \prod_i d^2 z_i e^{-\frac{2N}{\lambda}\bar{z}_i z_i} |\Delta(z)|^2 \left(e^{\sqrt{\frac{1}{2}}\sum_1^K (z_i + \bar{z}_i)}\right)}{\int \prod_i d^2 z_i e^{-\frac{2N}{\lambda}\bar{z}_i z_i} |\Delta(z)|^2} \tag{3.3}$$

To analyze this model, it is convenient to divide the eigenvalues into two sets, those that appear in the loop $z_1, \ldots, z_K$ and those which do not, $z_{K+1}, \ldots, z_N$.

Let us first analyze the case where K remains finite as $N \to \infty$, i.e. $\lim_{N\to\infty} \frac{K}{N} = 0$. In this case, the set of $N - K$ eigenvalues can be integrated out by solving for their saddle-point, which is again the circular droplet (2.8). The remaining K eigenvalues have an effective action which concentrates them within the circular droplet. Inside the droplet, the terms of order N in the effective action cancel, and to analyze the dynamics, one must take into account the $1/N$ corrections to the density, to get the effective action to order

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[3] For a general representation R with Young tableaux $(h_1, \ldots, h_N)$, the dimension is

$$\dim R = \prod_{1 \leq i < j \leq N} \frac{h_i - h_j + j - i}{j - i}$$

[4] Note that there is an interesting $K \to N - K$ duality in the integral (3.3). We find it by translating all of the eigenvalues $m_i \to m_i + \sqrt{\frac{\lambda}{8N}}$ and noting that the Vandermonde determinant is invariant under the simultaneous translation of all eigenvalues. Then, we transform $m_i \to -m_i$ to obtain

$$\langle W_{\mathcal{A}_K} \rangle = \langle W_{\mathcal{A}_{N-K}} \rangle \cdot e^{\left(2\frac{K}{N}-1\right)\frac{\lambda}{8}}$$

In particular, $\langle W_{\mathcal{A}_N} \rangle = e^{\frac{\lambda}{8}}$. Since $\mathcal{A}_N$ is trivial as a representation of SU(N), $\langle W_{\mathcal{A}_N} \rangle$ is equal to the circle Wilson loop in the non-interacting adjoint U(1) gauge theory.

one. A guess would be that the potential for eigenvalues inside the drop is simply flat so they move freely there. Then, our result for infinite N, finite K would be

$$\langle W_{A_K} \rangle = (-1)^{K+1} \langle W_\square \rangle^K \tag{3.4}$$

We shall see that this is borne out by a careful analysis below.

The effect of integrating out the $N - K$ eigenvalues is summarized by the joint probability distribution of remaining K eigenvalues [26]

$$\rho_N(z_1, \ldots, z_K) = \frac{N!}{(N-K)!} \frac{\int \prod_{a=K+1}^N d^2 z_a \, e^{-\frac{2N}{\lambda} \sum_{i=1}^N |z_i|^2} \prod_{i<j} |z_i - z_j|^2}{\int \prod_{i=1}^N d^2 z_a \, e^{-\frac{2N}{\lambda} \sum_{i=1}^N |z_i|^2} \prod_{i<j} |z_i - z_j|^2}$$

$$= \left(\frac{2}{\pi\lambda}\right)^K e^{-\sum_{i=1}^K \frac{2N}{\lambda}|z_i|^2} \det \left(\sum_{p=0}^{N-1} \frac{(\frac{2N}{\lambda} z_i \overline{z_j})^p}{p!} \right) \tag{3.5}$$

When $N \to \infty$, we must study the convergence of the sum in (3.5). It is easy to see that the single particle distribution converges to a positive constant if $|z_1| < \sqrt{\frac{\lambda}{2}}$ and zero if $|z_1|$ is outside of this radius [26]. This means that the eigenvalues are confined to a droplet. Then, it can be seen that the multi-particle distribution converges to a constant for all eigenvalues inside the droplet. In this way, we see that the eigenvalues are confined to the droplet and have a flat potential when inside the droplet.

In the large N limit with fixed K, ρ becomes

$$\rho(z_1, \ldots, z_K) = \left(\frac{2}{\pi\lambda}\right)^K e^{-\sum_{i=1}^K \frac{2N}{\lambda}|z_i|^2} \det(e^{\frac{2N}{\lambda} z_i \overline{z}_j}) \to \left(\frac{2}{\pi\lambda}\right)^K \quad |z_i| < \sqrt{\frac{\lambda}{2}} \tag{3.6}$$

It is easy to see that this produces (3.4). We will derive this result in a third way in appendix B.1.

In the case where $K \to \infty$ and the ratio K/N remains finite as $N \to \infty$, we again divide the eigenvalues into two subsets, so that one subset is those which couple directly to the Wilson loop and the other is those which do not. In this case, each subset contains a finite fraction of the eigenvalues. We introduce two densities,

$$\rho(z) = \frac{1}{K} \sum_{i=1}^K \delta(z - z_k) \;, \quad \hat\rho(z) = \frac{1}{N-K} \sum_{i=K+1}^N \delta(z - z_i) \quad \int d^2 z \rho(z) = 1 = \int d^2 z \hat\rho(z) \tag{3.7}$$

and the saddle point equations are

$$\frac{2}{\lambda} z = \frac{K}{N} \int d^2 z' \frac{\rho(z')}{\bar{z} - \bar{z}'} + \frac{N-K}{N} \int d^2 z' \frac{\hat\rho(z')}{\bar{z} - \bar{z}'} + \frac{1}{\sqrt{2}N} \quad z \in \text{support of } \rho$$

$$\frac{2}{\lambda} z = \frac{N-K}{N} \int d^2 z' \frac{\hat\rho(z')}{\bar{z} - \bar{z}'} + \frac{K}{N} \int d^2 z' \frac{\rho(z')}{\bar{z} - \bar{z}'} \quad z \in \text{support of } \hat\rho$$

In the large N limit, these would apparently be solved by any two functions such that

$$\frac{K}{N} \rho(z) + \frac{N-K}{N} \hat\rho(z) = \begin{cases} \frac{2}{\pi\lambda} & |z| < \sqrt{\frac{\lambda}{2}} \\ 0 & |z| > \sqrt{\frac{\lambda}{2}} \end{cases} \tag{3.8}$$

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They must add to form the circular droplet density in eq. (2.8), but otherwise any split of the disc distribution into two functions of the appropriate normalization will do. This is a highly degenerate solution and, again, the degeneracy must be split by corrections of order $1/N$. Here, it can be resolved by maximizing the integral

$$E_{\text{int}} = K \int d^2z \rho(z)(z + \bar z)$$

In the electrostatic analogy, this term is the interaction of a subset of the charged particles with an external electric field. A fraction K/N of the particles interact with this field and the remaining fraction $1 - K/N$ do not. This polarizes the droplet by pushing the K eigenvalues whose density is ρ to the right of the droplet and the other $N - K$ eigenvalues to the left (see figure 1b). The result is

$$\rho(z) = \frac{N}{K} \frac{2}{\pi \lambda} \, \Theta\left(\sqrt{\frac{\lambda}{2}} - |z|\right) \Theta(z + \bar z - \sqrt{2\lambda} \cos \theta) \tag{3.9}$$

$$\hat\rho(z) = \frac{N}{N - K} \frac{2}{\pi \lambda} \, \Theta\left(\sqrt{\frac{\lambda}{2}} - |z|\right) \Theta(\sqrt{2\lambda} \cos \theta - z - \bar z) \tag{3.10}$$

where Θ is the Heaviside function and the normalization of the densities in eq. (3.7) determines θ:

$$\theta - \frac{1}{2} \sin 2\theta = \pi \frac{K}{N}$$

Then the expectation value of the Wilson loop is given by

$$\langle W_{A_K} \rangle \;=\; \exp\left(\frac{1}{\sqrt{2}} \int d^2z \rho(z) \, (z + \bar z)\right) = \exp\left(\frac{2N\sqrt{\lambda}}{3\pi} \sin^3 \theta\right) \tag{3.11}$$

which agrees with the result of Yamaguchi [5] who argued that it matches the free energy of the 5-brane computed using the Born-Infeld action and who derived the same result from a Hermitian matrix model. This agreement between hermitian model analysis and the normal model can easily be seen directly by projecting the circular droplet to the real axis of z. Then the eigenvalue density along the real axis is given by the semi-circle law, which of course is a eigenvalue distribution of the Gaussian hermitian matrix model.

We summarize the results so far in figure 1. For the small winding, small representation loop, the eigenvalue which couples to the loop lives in a flat potential inside the disc. When the winding number gets large, it acts like an electric field which pulls the eigenvalue outside of the disc as depicted in figure 1a. This is associated with the D3-brane. In the case where there is a large multiplicity of small winding number loops, the eigenvalues still reside in the disc, but the loop polarizes them as in figure 1b. One can think of the line which divides the two subsets of eigenvalues in this case as the analog of a hole in the eigenvalue sea. In the case of a chiral primary operator, such a hole is also interpreted as a giant graviton which is extended in the S^5 direction. In the case of the Wilson loop, it seems to be more economical to interpret it as a 5-brane [6].

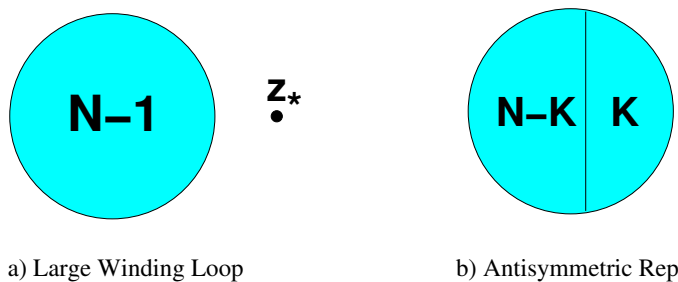a) Large Winding Loop   b) Antisymmetric Rep

Figure 1: Eigenvalue distribution in the normal matrix representation of Wilson loops. a) A large winding loop $W^{(k)}$ corresponds to a single eigenvalue sitting at $z = z_1^*$ outside the droplet. b) For a Wilson loop in the Kth antisymmetric representation A_K, the droplet is divided by a line $\mathrm{Re}z = \sqrt{\frac{\lambda}{2}} \cos\theta$ and K eigenvalues and $N - K$ eigenvalues are distributed on the right and left of this line, respectively.

3.1 Symmetric representation

There seem to be two competing descriptions of the D3-brane. In ref. [3] the claim was that the large wrapping number single trace Wilson loop is dual to a D3-brane. In [6] the Wilson loop was one in a representation consisting of a long single row Young diagram. In the following we will show that there are in fact two regimes. If λ is large enough, the two descriptions are in fact identical in the large N limit. If λ is lowered beyond a certain critical value ~ 5.5, there is a first order phase transition to a regime where the Wilson loop behaves like a source of D5-brane as described in ref. [5, 6].

We can also consider the Wilson loop in a representation which has Young diagram a single row ☐☐ ... ☐ with k boxes. The decomposition of the character in this representation into symmetric polynomials, for a diagonal matrix is

$$\mathrm{Tr}_{\text{☐☐...☐}}e^M = \frac{K!(N-1)!}{(N+K-1)!} \sum_{i_1 \leq i_j ... \leq i_k} e^{m_{i_1}+m_{i_2}+\cdots+m_{i_k}} \tag{3.12}$$

In order to use the formula (A.1), it is convenient to divide the sum in eq. (3.12) into special cases. Here, we will compare the two extreme such cases, first the set of terms where all of the indices are the same. This corresponds to a single large winding Wilson loop. Second, we will consider the case where all indices are different. This gives a contribution identical to the antisymmetric representation Wilson loop. In both cases, we will consider the case where k is large.

One would naively expect that the case with all indices different would dominate simply because it has a much higher multiplicity. In fact, if we were averaging the right-hand-side of (3.12) under an integral which was symmetric in the indices, the terms where all indices are different could be replaces by

$$\frac{N!}{k!(N-k)!}e^{m_1+m_2+\cdots+m_k}$$

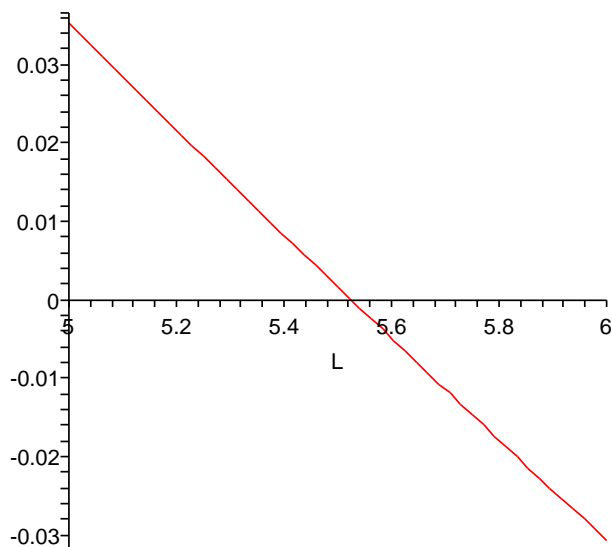The factor comes from the multiplicity of these terms.

Figure 2: This is the graph of the differences in the exponents in (3.13) minus (3.14) (divided by N) in the region where they cross. There is a first order phase transition where $\lambda \approx 5.5$. This figure is plotted for the case where $k = N/2$.

The other term of interest would be replaced by

$$Ne^{km_1}$$

and it occurs which a smaller factor.

The expectation value of the first of these is identical to the expectation value of the Wilson loop in an anti-symmetric representation which we have already computed. Taking into account the multiplicity factor and using eq. (3.11) we get

$$\exp\left(N\left(-\frac{k}{N}\ln\frac{k}{N} - (1-\frac{k}{N})\ln(1-\frac{k}{N}) + \frac{2\sqrt{\lambda}}{3\pi}\sin^3\theta\right)\right) \tag{3.13}$$

where $\theta - \frac{1}{2}\sin 2\theta = \pi\frac{k}{N}$.

This should be compared with the exponential in N part of the expectation value of Ne^{km_1} which we can read off from eq. (2.16), which gives

$$\exp\left(N\left(2\kappa\sqrt{\kappa^2+1} + 2\sinh^{-1}\kappa\right)\right) \;,\quad \kappa = \frac{k\sqrt{\lambda}}{4N} \tag{3.14}$$

To compare these, we choose $\frac{k}{N} = \frac{1}{2}$ and plot the difference between the exponents in eqs. (3.13) and (3.14) as a function of λ. The plot is depicted in figure 2. There we see a crossover between two regimes: when $\lambda < 5.5$ the case (3.13) dominates whereas for $\lambda > 5.5$ the case (3.14) dominates. We have not analyzed intermediate cases where sum but not all eignevalues are equal. We speculate that there is a phase transition between these two regimes. Seen from the point of view of the electrostatic analogy, in the first phase the eigenvalues which couple to the external electric field prefer to be independent.

In the second phase they prefer to sit on top of each other as one macroscopically charged particle. This occurs when λ is sufficiently large.

4. OPE of Wilson loops and chiral primary operators

So far the normal matrix model representation of Wilson loop and the droplet picture which emerges at large N are interesting technical tools, but all of the computations that we have outlined could also have been done, in principle, using the Hermitian Gaussian matrix model. However, when we consider the interaction between Wilson loop and chiral primary operator, the normal matrix model representation provides us with new insight.

As discussed in ref. [14], a 1/2-BPS chiral primary operator has a dual description in supergravity which is specified by the shape of a droplet. If we think of the chiral primary operator as a state, the characteristic of that state which is visible in geometry is the shape of the droplet that is associated with it.

The disc-shaped droplet that we have discussed so far is the ground state. The state of IIB string theory corresponding to it is the ground state on the $AdS_5 \times S^5$ background. Small chiral primaries correspond to the same background with some gravitons excited. Larger chiral primaries are giant gravitons or, if they are large enough, they modify the geometry and the state is string theory on a different background. In the large N limit these states are all characterized by droplets which, now are different from the ground state disc droplet.

The vacuum expectation value of the Wilson loop operator measures the vacuum matrix element and gives the content of the unit operator in the loop. We can consider some other matrix elements as well, such as the matrix element between the vacuum and the state which is created by a chiral primary operator. This matrix element is of course given in the integral in eq. 1.10. In the language of eigenvalues, it measures the ability of the Wilson loop operator to modify the circular droplet which characterizes the ground state so that it fits with the droplet for an excited state, the chiral primary.

Let us consider the OPE between the Wilson loop and the chiral primary operator $\mathcal{O}_k = \frac{1}{\sqrt{kN^k}}\mathrm{Tr}Z^k$. In the next section, we shall prove that it is given by a normal matrix model

$$\langle W_\square \mathcal{O}_k \rangle = 2^{-\frac{k}{2}} \frac{1}{Z_N} \int_{[z,\overline{z}]=0} [d^2z] e^{-\mathrm{Tr}(z\overline{z})} \frac{1}{N} \mathrm{Tr}\, e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z+\overline{z}) - \frac{\lambda}{8N}} \frac{1}{\sqrt{kN^k}} \mathrm{Tr}z^k \qquad (4.1)$$

The normalization of $\mathcal{O}_k$ is chosen so that the two-point function is normalized to unity in the large N limit. In the eigenvalue basis, (4.1) is written as

$$\langle W_\square \mathcal{O}_k \rangle = 2^{-\frac{k}{2}} \frac{e^{-\frac{\lambda}{8N}}}{Z_N} \int \prod_{i=1}^{N} d^2z_i e^{-|z_i|^2} |\Delta(z)|^2 \frac{1}{N\sqrt{kN^k}} \sum_{i,j=1}^{N} e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}(z_i+\overline{z}_j)} z_j^k \qquad (4.2)$$

This is evaluated exactly using the joint probability distribution (3.5).[5]

---

[5]This is a more subtle application of the joint probability distribution since it probes its next-to-leading

As discussed in the previous section, we can integrate out the radial coordinate $|z_i|$ and rewrite this OPE coefficient as a contour integral. However, to see the relation to the droplet picture, we find it useful to integrate out the angular part of z_i. After some computation, we find the exact finite N expression for the normal matrix integral (4.1)

$$\langle W_\square \mathcal{O}_k \rangle = \frac{e^{-\frac{\lambda}{8N}}}{N\sqrt{kN^k}} \int_0^\infty d|z|^2 \, e^{-|z|^2} \sum_{i=1}^k \frac{|z|^{2(N-i)}}{(N-i)!} |z|^k I_k \left(\sqrt{\frac{\lambda}{N}} |z| \right) \qquad (4.3)$$

In the large N limit, the $|z|$-integral is localized to the saddle point $|z| = \sqrt{N}$. Physically, this corresponds to the localization of the eigenvalue to the boundary of circular droplet. This is consistent with the picture that the chiral primary $\mathcal{O}_k$ with $k \ll N$ corresponds to a graviton which in turn is represented by a small ripple of the Fermi surface. Finally, the OPE coefficient at large N is found to be

$$\langle W_\square \mathcal{O}_k \rangle = \frac{1}{N} 2^{-\frac{k}{2}} \sqrt{k} I_k(\sqrt{\lambda}) \qquad (4.4)$$

which produces the expression in eq. (1.9) which was originally derived in ref. [24].

In the derivation of (4.4) we have assumed that $k \ll N$. One interesting problem where this is not the case is the BMN limit $k = J \sim \sqrt{N}$. In this limit, the integral (4.3) behaves differently. To take the BMN limit, it is useful to rewrite the integral (4.3) as a sum of Laguerre polynomials

$$\langle W_\square \mathcal{O}_k \rangle = \frac{1}{N\sqrt{kN^k}} \left(\frac{\lambda}{4N} \right)^{\frac{k}{2}} e^{\frac{\lambda}{8N}} \sum_{i=1}^k L_{N-i}^k \left(-\frac{\lambda}{4N} \right) \qquad (4.5)$$

Using the contour integral representation of Laguerre polynomials

$$a^k L_m^k(-a^2) = \oint \frac{dw}{2\pi i} e^{aw} w^{k-1} (1 + aw^{-1})^{m+k} \qquad (4.6)$$

the summation over i in (4.5) can be easily performed. In this way, we arrive at the contour integral representation of the normal matrix integral (4.1)

$$\langle W_\square \mathcal{O}_J \rangle = \frac{2}{\sqrt{J\lambda}} e^{\frac{\lambda}{8N}} \oint \frac{dw}{2\pi i} w^J e^{\frac{\sqrt{\lambda}}{2}w} \left(1 + \frac{\sqrt{\lambda}}{2Nw} \right)^N \left[\left(1 + \frac{\sqrt{\lambda}}{2Nw} \right)^J - 1 \right] \qquad (4.7)$$

In this expression, it is easy to take the BMN limit

$$N, J \to \infty \quad g_2 = \frac{J^2}{N}, \lambda' = \frac{\lambda}{J^2} \quad \text{fixed} \qquad (4.8)$$

In this limit the integral (4.7) becomes a non-trivial function of g_2 and λ'

$$\langle W_\square \mathcal{O}_J \rangle \sim \frac{2}{\sqrt{J^3 \lambda'}} e^{\frac{g_2 \lambda'}{8}} \oint \frac{dw}{2\pi i} w^J e^{\frac{J\sqrt{\lambda'}}{2}(w+w^{-1})} \left(e^{\frac{g_2\sqrt{\lambda'}}{2w}} - 1 \right) \qquad (4.9)$$

order behavior in $1/N$. The naive leading order in large N, which would be the factorized product of two traces, vanishes after the integral over phases of z. This means that we must take into account the next-to-leading order. We do this by integrating the exact formula to get (4.4) and then taking the large N limit.

This integral is evaluated around the saddle point of the term proportional to J

$$\partial_w \left[\frac{\sqrt{\lambda'}}{2}(w + w^{-1}) + \log w \right] = 0 \quad \longrightarrow \quad w_* = \sqrt{1 + \frac{1}{\lambda'}} - \frac{1}{\sqrt{\lambda'}} \tag{4.10}$$

Then we find

$$\langle W_\square \mathcal{O}_J \rangle = \frac{1}{N} \mathcal{A} e^{\mathcal{S}} \tag{4.11}$$

where $\mathcal{S}$ and $\mathcal{A}$ are given by

$$\mathcal{S} = \sqrt{\lambda} \left[\sqrt{1 + \frac{1}{\lambda'}} + \frac{1}{\sqrt{\lambda'}} \log \left(\sqrt{1 + \frac{1}{\lambda'}} - \frac{1}{\sqrt{\lambda'}} \right) \right]$$

$$\mathcal{A} = \frac{1}{\sqrt{2\pi}} (1 + \lambda'^{-1})^{-\frac{1}{4}} e^{\frac{g_2}{8}(1+\sqrt{1+\lambda'})^2} \frac{\sinh\left[\frac{g_2}{4}(1+\sqrt{1+\lambda'})\right]}{\frac{g_2}{4}(1+\sqrt{1+\lambda'})} \tag{4.12}$$

Since g_2 is the genus-counting parameter in the BMN limit, this expression gives an all-genus result of OPE between a single winding Wilson loop $W_\square$ and the 1/2 BPS BMN operator $\text{Tr}Z^J$. The expression of $\mathcal{S}$ agrees with the result of [37].

On the other hand, the coupling to the large winding loop $W^{(w)}$ ($w \gg 1$) behaves very differently. As we saw above, the saddle point of z-integral is at $z = z_1^*$ (2.15). Therefore, the OPE coefficient behaves as

$$\langle W_\square^{(w)} \mathcal{O}_k \rangle \sim e^{2N\kappa\sqrt{\kappa^2+1}+2(N+k)\sinh^{-1}\kappa} \tag{4.13}$$

We can write down the OPE between the Wilson loop in the Kth anti-symmetric representation and the chiral primary $\mathcal{O}_J$ in the form of the contour integral as before

$$\langle W_{A_K} \mathcal{O}_J \rangle = \frac{KN!}{(N-K)!\sqrt{J}} \oint \prod_{i=1}^{K} \frac{dz_i}{2\pi i} e^{\frac{\sqrt{\lambda}}{2} z_i} \left(z_1 + \frac{\sqrt{\lambda}}{2N} \right)^J \det A \tag{4.14}$$

where $K \times K$ matrix A is given by

$$A_{i1} = \frac{(z_i + \frac{\sqrt{\lambda}}{2N})^{N-J}}{z_1^N} \cdot \frac{(z_i + \frac{\sqrt{\lambda}}{2N})^J - z_1^J}{z_i + \frac{\sqrt{\lambda}}{2N} - z_1}$$

$$A_{ij} = \frac{(z_i + \frac{\sqrt{\lambda}}{2N})^N z_j^{-N} - 1}{z_i + \frac{\sqrt{\lambda}}{2N} - z_j} \quad (j \neq 1) \tag{4.15}$$

When K is of order 1 and N is sent to infinity, the dominant contribution comes from the diagonal elements of the matrix A

$$\langle W_{A_K} \mathcal{O}_J \rangle = \frac{K}{N^K \sqrt{J}} \oint \prod_{i=1}^{K} \frac{dz_i}{2\pi i} e^{\frac{\sqrt{\lambda}}{2} z_i} \left(z_1 + \frac{\sqrt{\lambda}}{2N} \right)^J \prod_{i=1}^{K} A_{ii}$$

$$\sim K \left(\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \right)^{K-1} \frac{1}{N} \sqrt{J} I_J(\sqrt{\lambda})$$

$$= K \langle W_\square \rangle^{K-1} \langle W_\square \mathcal{O}_J \rangle \tag{4.16}$$

Namely, the OPE coefficient factorizes when K is small.

The more natural basis of chiral primary is given by the Schur polynomial of $Z = \mathrm{diag}(z_1, \ldots, z_N)$

$$\mathcal{O}_R = \frac{\det(z_i^{h_j+N-j})}{\det(z_i^{N-j})} = \frac{\det(z_i^{h_j+N-j})}{\Delta(z)} \tag{4.17}$$

Here $h_1 \geq h_2 \cdots \geq h_N$ are the row length of the Young diagram R. Using the normal matrix model, one can easily show that the two-point function of $\mathcal{O}_R$ is diagonal in R. Similarly, one can show that the OPE between $W_\square$ and the chiral primary $\mathcal{O}_R$ is non-zero only if R is a single hook $i.e.$, $h_1 \geq 1, h_2 = \cdots = h_k = 1, h_{k+1} = \cdots = h_N = 0$

$$\langle W_\square \mathcal{O}_R \rangle = 0 \quad \text{if} \quad R \neq \text{⌐□□□□} \tag{4.18}$$

Explicitly, the OPE is given by

$$\langle W_\square \mathcal{O}_{\text{⌐□□□}} \rangle = \frac{(-1)^{k-1}}{N} N^{\frac{|R|}{2}} \oint \frac{dz}{2\pi i z} e^{\frac{\sqrt{\lambda}}{2} z} z^{|R|} \left(1 + \frac{\sqrt{\lambda}}{2Nz} \right)^{N+h_1-1}$$

$$\sim \frac{(-1)^{k-1}}{N} (N+h_1)^{\frac{|R|}{2}} I_{|R|}\left(\sqrt{g_{\mathrm{YM}}^2 (N+h_1)} \right) \tag{4.19}$$

where k is the length of the first column and $|R| = h_1 + k - 1$ is the number of boxes. It is interesting that N is effectively replaced by $N + h_1$.

For a more general Wilson loop W_R, the multi-hook chiral primary can couple to it. The computation becomes harder as the number of hooks increases. Therefore, we restrict ourselves to the case of single-hook chiral primary. The OPE between Wilson loop in the Kth anti-symmetric representation and the single-hook chiral primary is easily obtained as

$$\langle W_{A_K} \mathcal{O}_{\text{⌐□□□}} \rangle = \oint \prod_{i=1}^{K} \frac{dz_i}{2\pi i} e^{\frac{\sqrt{\lambda}}{2} z_i} z_1^{-1} \left(z_1 + \frac{\sqrt{\lambda}}{2N} \right)^{h_1+k-1} \det B \tag{4.20}$$

where $K \times K$ matrix B is given by

$$B_{i1} = \left(z_i + \frac{\sqrt{\lambda}}{2N} \right)^{N-k} z_1^{-N+k}$$

$$B_{ij} = \frac{(z_i + \frac{\sqrt{\lambda}}{2N})^N z_j^{-N} - 1}{z_i + \frac{\sqrt{\lambda}}{2N} - z_j} \quad (j \neq 1) \tag{4.21}$$

Again, when K is of order 1 the dominant contribution comes from the diagonal part of B. In this case one finds

$$\langle W_{A_K} \mathcal{O}_{\text{⌐□□□}} \rangle \sim K \langle W_\square \rangle^{K-1} \langle W_\square \mathcal{O}_{\text{⌐□□□}} \rangle \tag{4.22}$$

As in the case of the Wilson loop correlator $\langle W_{A_K} \rangle$, we expect that the OPE coefficient $\langle W_{A_K} \mathcal{O}_R \rangle$ fails to factorize when $K \sim N$. We leave this analysis as an interesting future problem.

5. Discussion

In this paper, we have shown that the expectation values of circular Wilson loops and the OPE between Wilson loops and chiral primaries both can be written as normal matrix models. This representation as a normal matrix model is exact for some representations of Wilson loop and it is approximate for some other cases.

This normal matrix model gives a unified description of 1/2-BPS chiral primary operators and the 1/2-BPS Wilson loops. In particular, there is a nice analogy between dual giant graviton and large winding Wilson loop. Namely, the dual giant graviton wrapped around S^3 inside AdS_5 is identified as a particle outside the droplet, whereas the large winding Wilson loop also corresponds to an isolated eigenvalue outside the droplet. Moreover it has been proposed that the object dual to a large winding loop is a D3-brane with world-volume $AdS_2 \times S^2$ and extended into AdS_5. The direction that the dual object blows up into (i.e., AdS_5 direction) is the same as the case of the giant graviton.

In the case of giant graviton wrapped around an S^3 inside S^5, the droplet picture is a hole inside the droplet and the corresponding operator is a Schur polynomial of anti-symmetric representation. The corresponding Wilson loop is in an anti-symmetric representation of gauge group, and the dual object is a D5-brane with world-volume $AdS_2 \times S^4$. The S^4 part of the world-volume is inside S^5, so the blowing-up direction (S^5 in this case) is again common for both the chiral primary and the Wilson loop. However, the droplet picture of an anti-symmetric Wilson loop is a bit different from the giant graviton: it corresponds to a line dividing the droplet. Still there is a similarity between the droplet picture of giant graviton and anti-symmetric Wilson loop, namely they are both represented by objects inside the droplet. It would be nice to push this analogy further.

We have noted that, when λ is large, a symmetric representation Wilson loop is practically identical to a large winding number single trace Wilson loop. This explains why they can both be associated with the D3-brane. We also observe that, there is another phase which is stable for smaller λ and a phase transition at $\lambda \approx 5.5$. In that phase, the free energy resembles the D5-brane. It is interesting that this case has a D5-brane blowing up into the AdS_5 direction.

We also considered the OPE between Wilson loop and chiral primary. At least technically, our normal matrix model gives an efficient way to compute this OPE. We found a peculiar selection rule for the OPE. For example, the Wilson loop in the fundamental representation couples only to the chiral primary with single hook representation. Similarly, the Kth anti-symmetric loop couples to the chiral primary with at most K hooks. We should emphasize that this selection rule is exact. The meaning of selection rule in the bulk AdS side is not clear to us. It would be nice to understand this from the string theory side.

Recently, the supergravity dual of the 1/2-BPS Wilson loop has been constructed [4, 10]. The solution is characterized by a boundary value problem in two-dimensions, and the one-dimensional boundary line is divided into black and white regions. It is suggested [4] that these regions corresponds to the eigenvalue distribution of a hermitian matrix model. It would be nice to clarify this relation, if any. Our normal matrix model description seems

to suggest the existence of M-theory version of the supergravity dual, where the boundary region is two-dimensional and the boundary data are droplets.

We mainly considered the Wilson loop in the representation whose Young diagram has a number of boxes of order N. We expect that when the number of boxes becomes of order N^2, the dual object can no longer be regarded as a probe of the fixed background $AdS_5 \times S^5$, rather the geometry is significantly deformed. In our droplet picture, the effect of such big Wilson loop will be to deform the shape of the circular droplet. For example, the anti-symmetric representation of the large winding loop $\langle W_{A_K}^{(w)} \rangle$ might correspond to two disconnected droplets when $Kw \sim N^2$. It would be interesting to find the eigenvalue distribution for this kind of large Wilson loop.

A. An integral formula

In the following, we shall prove the integral formula

$$\frac{\int \prod_{i=1}^N dx_i e^{-x_i^2} \Delta(x)^2 \prod_{i=1}^N e^{k_i x_i}}{\int \prod_{i=1}^N dx_i e^{-x_i^2} \Delta(x)^2} = \frac{\int \prod_{i=1}^N d^2 z_i e^{-|z_i|^2} |\Delta(z)|^2 \prod_{i=1}^N e^{\frac{k_i}{\sqrt{2}}(z_i + \overline{z}_i) - \frac{1}{4}k_i^2}}{\int \prod_{i=1}^N d^2 z_i e^{-|z_i|^2} |\Delta(z)|^2} \quad (A.1)$$

which relates the expectation value of a general exponential in the Hermitian matrix model to one in the normal matrix model.

The matrix eigenvalue integral on the left-hand-side of eq. (A.1) is written as

$$\frac{\int \prod_{i=1}^N dx_i e^{-x_i^2} \Delta(x)^2 \prod_{i=1}^N e^{k_i x_i}}{\int \prod_{i=1}^N dx_i e^{-x_i^2} \Delta(x)^2} = \frac{1}{N!} \int \prod_{i=1}^N dx_i \Psi(x_1, \ldots, x_N)^2 \prod_i \left(e^{k_i x_i} \right) \quad (A.2)$$

where $\Psi(x_1, \ldots, x_N)$ is the Slater determinant which is the wave-function for the N fermions in a harmonic potential

$$\Psi(x_1, \ldots, x_N) = \det_{1 \leq i,j \leq N} (\psi_{i-1}(x_j))$$
$$\psi_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-\frac{1}{2}x^2} \quad (A.3)$$

$H_n(x) = e^{x^2}(-\partial_x)^n e^{-x^2}$ is the Hermite polynomial. In (A.2), we used the relation

$$\frac{1}{N!} \Psi(x_1, \ldots, x_N)^2 = \frac{1}{Z_H} e^{-\sum_{i=1}^N x_i^2} \Delta(x)^2 \quad (A.4)$$

where Z_H is the partition function of hermitian matrix model

$$Z_H = \int \prod_{i=1}^N dx_i e^{-x_i^2} \Delta(x)^2 = 2^{-\frac{N^2}{2}} (2\pi)^{\frac{N}{2}} \prod_{n=1}^N n! \quad (A.5)$$

We can further rewrite (A.2) in the operator formalism using the N free oscillators $a_i, a_i^\dagger$ ($[a_i, a_j^\dagger] = \delta_{ij}$). Using the relation $\widehat{x}_i = \frac{(a_i + a_i^\dagger)}{\sqrt{2}}$, (A.2) becomes

$$\frac{\int \prod_{i=1}^N dx_i e^{-x_i^2} \Delta(x)^2 \prod_{i=1}^N e^{k_i x_i}}{\int \prod_{i=1}^N dx_i e^{-x_i^2} \Delta(x)^2} = \frac{1}{N!} \langle \Psi | \prod_{i=1}^N e^{\frac{1}{\sqrt{2}} k_i (a_i + a_i^\dagger)} | \Psi \rangle \quad (A.6)$$

where $|\Psi\rangle$ is the anti-symmetric N-particle state

$$
\begin{aligned}
|\Psi\rangle &= |0\rangle \wedge |1\rangle \wedge \cdots \wedge |N-1\rangle \\
&= \sum_{\sigma \in S_N} (-1)^{\deg \sigma} |\sigma(1)-1\rangle \otimes \cdots \otimes |\sigma(N)-1\rangle \quad \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N
\end{aligned} \tag{A.7}
$$

and $|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$ is the nth excited state. Here $\mathcal{H}_i$ denotes the Fock space of ith oscillator and the operators $a_i, a_i^\dagger$ act on $\mathcal{H}_i$. The state $|\Psi\rangle$ represents a Fermi sea of N fermions in a harmonic potential. To evaluate (A.6), we need to know the overlap $\langle n | e^{k\widehat{x}} | m\rangle$. In order to rewrite the hermitian matrix model to normal matrix model, the key step is to insert the completeness relation of the coherent state $|z\rangle = e^{za^\dagger}|0\rangle$

$$
\mathbf{1} = \frac{1}{\pi} \int d^2 z \, e^{-|z|^2} |z\rangle\langle z| \tag{A.8}
$$

Then the matrix element $\langle n | e^{k\widehat{x}} | m\rangle$ is written as

$$
\begin{aligned}
\langle n | e^{k\frac{a+a^\dagger}{\sqrt{2}}} | m\rangle &= e^{-\frac{1}{4}k^2} \langle n | e^{\frac{k}{\sqrt{2}}a} e^{\frac{k}{\sqrt{2}}a^\dagger} | m\rangle \\
&= e^{-\frac{1}{4}k^2} \frac{1}{\pi} \int d^2 z \, e^{-|z|^2} \langle n | e^{\frac{k}{\sqrt{2}}a} | z\rangle \langle z | e^{\frac{k}{\sqrt{2}}a^\dagger} | m\rangle \\
&= \frac{1}{\pi\sqrt{n!m!}} \int d^2 z \, e^{-|z|^2 + \frac{k}{\sqrt{2}}(z+\overline{z}) - \frac{1}{4}k^2} z^n \overline{z}^m
\end{aligned} \tag{A.9}
$$

Let us consider a term appearing in (A.2) of the form

$$
\int \prod_{i=1}^N dx_i \, \Psi(x_1, \ldots, x_N)^2 \prod_{i=1}^N e^{k_i x_i} = \langle \Psi | \prod_{i=1}^N e^{k_i \widehat{x}_i} | \Psi\rangle \tag{A.10}
$$

Using the relation (A.9), this is written as

$$
\begin{aligned}
\frac{1}{N!} \langle \Psi | \prod_{i=1}^N e^{k_i \widehat{x}_i} | \Psi\rangle &= \frac{1}{N!} \sum_{\sigma, \tau \in S_N} (-1)^{\sigma+\tau} \prod_{i=1}^N \langle \sigma(i)-1 | e^{k_i \widehat{x}_i} | \tau(i)-1\rangle \\
&= \frac{1}{Z_N} \sum_{\sigma, \tau \in S_N} (-1)^{\sigma+\tau} \prod_{i=1}^N \int d^2 z_i \, e^{-|z_i|^2 + \frac{k_i}{\sqrt{2}}(z_i+\overline{z}_i) - \frac{1}{4}k_i^2} z_i^{\sigma(i)-1} \overline{z}_i^{\tau(i)-1} \\
&= \frac{1}{Z_N} \prod_{i=1}^N \int d^2 z_i \, e^{-|z_i|^2 + \frac{k_i}{\sqrt{2}}(z_i+\overline{z}_i) - \frac{1}{4}k_i^2} \Delta(z) \Delta(\overline{z})
\end{aligned} \tag{A.11}
$$

where Z_N is the partition function of normal matrix model

$$
Z_N = \int \prod_{i=1}^N d^2 z_i \, e^{-|z_i|^2} |\Delta(z)|^2 = \pi^N \prod_{n=1}^N n! \tag{A.12}
$$

From (A.4) and (A.11), we find the identity between the Gaussian hermitian matrix model and the Gaussian normal matrix model

$$
\frac{1}{Z_H} \int \prod_{i=1}^N dx_i \, e^{-x_i^2} \Delta(x)^2 \prod_{i=1}^N e^{k_i x_i} = \frac{1}{Z_N} \int \prod_{i=1}^N d^2 z_i \, e^{-|z_i|^2} |\Delta(z)|^2 \prod_{i=1}^N e^{\frac{k_i}{\sqrt{2}}(z_i+\overline{z}_i) - \frac{1}{4}k_i^2} \tag{A.13}
$$

We should stress that the relation (A.13) is exact.

B. Contour integral representation

In this appendix, we rederive (3.4) by directly integrating out the $N - K$ eigenvalues using the joint probability distribution (3.5).

$$\langle W_{A_K} \rangle = \frac{N!}{(N-K)!} e^{-\frac{K\lambda}{8N}} \prod_{i=1}^{K} \int \frac{d^2 z_i}{\pi} e^{-|z_i|^2 + \frac{1}{2}\sqrt{\frac{\lambda}{N}}(z_i + \overline{z}_i)} \det \left(\sum_{p=0}^{N} \frac{(z_i \overline{z}_j)^p}{p!} \right) \qquad \text{(B.1)}$$

Here and the following section, we use the rescaled variables $z_i^{\text{new}} = \sqrt{\frac{2N}{\lambda}} z_i^{\text{old}}$ to make the Gaussian part $|z_i^{\text{new}}|^2$ simple.

As we will explain in the next subsection of this appendix, we can integrate out the radial coordinate of z_i and the the angular integral reduces to a contour integral.

$$\langle W_{A_K} \rangle =$$

$$= \frac{N!}{(N-K)!} e^{-\frac{K\lambda}{8N}} \prod_{i=1}^{K} \int \frac{d^2 z_i}{\pi} e^{-|z_i|^2 + \frac{1}{2}\sqrt{\frac{\lambda}{N}}(z_i + \overline{z}_i)} \sum_{p_1,\ldots,p_K=0}^{N-1} \sum_{\sigma \in S_K} (-1)^\sigma \prod_{i=1}^{K} \frac{z_i^{p_{\sigma(i)}} \overline{z}_i^{p_i}}{p_i!}$$

$$= \frac{N!}{(N-K)!} e^{\frac{K\lambda}{8N}} \prod_{i=1}^{K} \oint \frac{dz_i}{2\pi i z_i} e^{\frac{1}{2}\sqrt{\frac{\lambda}{N}}z_i} \sum_{p_1,\ldots,p_K=0}^{N-1} \sum_{\sigma \in S_K} (-1)^\sigma \prod_{i=1}^{K} \frac{(z_i + \frac{1}{2}\sqrt{\frac{\lambda}{N}})^{p_{\sigma(i)}}}{z_i^{p_i}}$$

$$= \frac{N!}{(N-K)!} e^{\frac{K\lambda}{8N}} \prod_{i=1}^{K} \oint \frac{dz_i}{2\pi i z_i} e^{\frac{1}{2}\sqrt{\lambda}z_i} \sum_{p_1,\ldots,p_K=0}^{N-1} \sum_{\sigma \in S_K} (-1)^\sigma \prod_{i=1}^{K} (z_{\sigma(i)} + \frac{\lambda}{2N})^{p_i} z_i^{-p_i}$$

$$= \frac{N!}{(N-K)!} e^{\frac{K\lambda}{8N}} \prod_{i=1}^{K} \oint \frac{dz_i}{2\pi i} e^{\frac{1}{2}\sqrt{\lambda}z_i} \det_{1 \le i,j \le K} \left(\frac{(z_i + \frac{\sqrt{\lambda}}{2N})^N z_j^{-N} - 1}{z_i - z_j + \frac{\sqrt{\lambda}}{2N}} \right) \qquad \text{(B.2)}$$

We should stress that this is an exact expression for anti-symmetric representation at finite N and K. In the large N limit with finite K, the dominant contribution comes from the diagonal elements of the matrix inside the determinant. In this way, we see that the integral factorizes

$$\langle W_{A_K} \rangle \sim \frac{N! N^K}{(N-K)!} \left(\frac{2}{\sqrt{\lambda}} \right)^K \prod_{i=1}^{K} \oint \frac{dz_i}{2\pi i} e^{\frac{1}{2}\sqrt{\lambda}z_i} \left[\left(1 + \frac{\sqrt{\lambda}}{2N z_i} \right)^N - 1 \right]$$

$$\sim \left(\frac{2}{\sqrt{\lambda}} \right)^K \prod_{i=1}^{K} \oint \frac{dz_i}{2\pi i} e^{\frac{1}{2}\sqrt{\lambda}(z_i + z_i^{-1})} = \left(\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \right)^K = \langle W_\square \rangle^K \qquad \text{(B.3)}$$

Although (B.2) is an exact formula, it seems difficult to take the large N and K limit with fixed K/N in this expression.

B.1 Integrating out the radial coordinate $|z|$ in (A.1)

As promised above, let us consider the integration of the radial coordinates $|z_i|$ of the normal matrix eigenvalues z_i. The remaining angular integral can be written as a contour integral.

In (A.9), the matrix element of $e^{\frac{k}{\sqrt{2}}(a+a^\dagger)}$ was written as the integral over z in the coherent state $|z\rangle$. We write this as an integral over $|z|$ and $\arg(z) = \theta$

$$
\begin{aligned}
\langle n|e^{k\widehat{x}}|m\rangle &= \frac{1}{\sqrt{n!m!}} \int_0^\infty d|z|^2 \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-|z|^2 + \frac{k}{\sqrt{2}}|z|(e^{i\theta}+e^{-i\theta}) - \frac{1}{4}k^2}|z|^{n+m}e^{i\theta(n-m)} \\
&= \frac{1}{\sqrt{n!m!}} \int_0^\infty dt \oint \frac{dz}{2\pi i z} e^{-t + \frac{k}{\sqrt{2}}\sqrt{t}(z+z^{-1}) - \frac{1}{4}k^2} t^{\frac{n+m}{2}} z^{n-m}
\end{aligned} \tag{B.4}
$$

where we introduced the variable $t = |z|^2$ and rewrote the θ-integral as a contour integral. After rescaling $z \to t^{-\frac{1}{2}}z$, the t-integral can be performed easily

$$
\begin{aligned}
\langle n|e^{k\widehat{x}}|m\rangle &= \frac{1}{\sqrt{n!m!}} \int_0^\infty dt \oint \frac{dz}{2\pi i z} e^{-t + \frac{k}{\sqrt{2}}z + \frac{k}{\sqrt{2}}tz^{-1} - \frac{1}{4}k^2} t^m z^{n-m} \\
&= \sqrt{\frac{m!}{n!}} \oint \frac{dz}{2\pi i} e^{\frac{k}{\sqrt{2}}z - \frac{1}{4}k^2} \frac{z^m}{(z - \frac{k}{\sqrt{2}})^{m+1}} \\
&= \sqrt{\frac{m!}{n!}} \oint \frac{dz}{2\pi i} e^{\frac{k}{\sqrt{2}}z + \frac{1}{4}k^2} \frac{(z + \frac{k}{\sqrt{2}})^n}{z^{m+1}}
\end{aligned} \tag{B.5}
$$

In the last step, we have shifted $z \to z + \frac{k}{\sqrt{2}}$. Using this expression, our key formula (A.11) is written as a contour integral

$$
\begin{aligned}
\frac{1}{N!}\langle\Psi| \prod_{i=1}^N e^{k_i\widehat{x}_i}|\Psi\rangle &= \\
&= \frac{1}{N!} \sum_{\sigma,\tau \in S_N} (-1)^{\sigma+\tau} \prod_{a=1}^N \oint \frac{dz_a}{2\pi i z_a} e^{\frac{k_a}{\sqrt{2}}z_a + \frac{1}{4}k_a^2} \frac{(z_a + \frac{k_a}{\sqrt{2}})^{\sigma(a)-1}}{z_a^{\tau(a)-1}} \\
&= \frac{1}{N!} \prod_{a=1}^N \oint \frac{dz_a}{2\pi i z_a} e^{\frac{k_a}{\sqrt{2}}z_a + \frac{1}{4}k_a^2} \prod_{1\le i<j\le N} \left(z_i + \frac{k_i}{\sqrt{2}} - z_j - \frac{k_j}{\sqrt{2}} \right) (z_i^{-1} - z_j^{-1})
\end{aligned} \tag{B.6}
$$

This almost looks like a unitary matrix model. For example, when

$$
k_1 = \cdots = k_K = \ell, \quad k_{K+1} = \cdots = k_K = 0 \tag{B.7}
$$

(B.6) becomes a coupled system of $U(K)$ and $U(N-K)$ unitary matrix model

$$
\begin{aligned}
\frac{1}{N!}\langle\Psi| \prod_{i=1}^K e^{\ell\widehat{x}_i}|\Psi\rangle &= \frac{1}{N!} e^{\frac{1}{4}K\ell^2} \prod_{i=1}^K \int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{\frac{\ell}{\sqrt{2}}e^{i\theta_i}} |\Delta(\theta)|^2 \prod_{j=1}^{N-K} \int_0^{2\pi} \frac{d\phi_j}{2\pi} |\Delta(\phi)|^2 \\
&\quad \times \prod_{i=1}^K \prod_{j=1}^{N-K} \left(e^{i\theta_i} + \frac{\ell}{\sqrt{2}} - e^{i\phi_j} \right) (e^{-i\theta_i} - e^{-i\phi_j})
\end{aligned} \tag{B.8}
$$

C. Normal matrix representation of OPE between Wilson loop and chiral primary

The OPE between Wilson loop and chiral primary is given by a complex matrix model

$$
\int [d^2z] e^{-\text{Tr}(z\overline{z})} \frac{1}{N} \text{Tr}\, e^{\frac{1}{2}(z+\overline{z})} \mathcal{O}_R(\overline{z}) \tag{C.1}
$$

Here, for notational simplicity, we have set $\frac{\lambda}{2N} = 1$. This parameter can easily be restored by the rescaling of the matrix variable. Eq. (C.1) can also written as integrals over the real and imaginary parts of the complex matrix z,

$$\int [dX dY] e^{-\mathrm{Tr}(X^2 + Y^2)} \frac{1}{N} \mathrm{Tr}\, e^X \mathcal{O}_R(X - iY) \tag{C.2}$$

Then we can formally do the integral over the imaginary part Y to get

$$\int [dX dY] e^{-\mathrm{Tr}(X^2 + Y^2)} \frac{1}{N} \mathrm{Tr}\, e^X \, e^{-i\mathrm{Tr} Y^T \frac{\partial}{\partial X}} \mathcal{O}_R(X) =$$
$$= \int [dX] e^{-\mathrm{Tr} X^2} \frac{1}{N} \mathrm{Tr}\, e^X \, e^{-\frac{1}{4}\mathrm{Tr}\left(\frac{\partial}{\partial X}\right)^2} \mathcal{O}_R(X) \tag{C.3}$$

Note that we have taken into account the Wick contractions of powers of Y which had appeared in the chiral primary by operating with a derivative operator. Next we can diagonalize the Hermitian matrix X by conjugating it with a unitary matrix U

$$X = U \mathrm{diag}(x_1, \ldots, x_N) U^{-1} \tag{C.4}$$

In the eigenvalue basis, the matrix Laplacian is given by

$$\mathrm{Tr}\left(\frac{\partial}{\partial X}\right)^2 = \frac{1}{\Delta(x)} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \Delta(x) + \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \frac{\partial^2}{\partial \omega_{ij} \partial \omega_{ji}} \tag{C.5}$$

where $\omega = U^{-1} dU$ is the "angular" part. Since $\mathcal{O}_R(X)$ is independent of U, the Calogero-type term in the matrix Laplacian does not contribute. Therefore, the matrix integral reduces to the eigenvalue model

$$\int \prod_{i=1}^{N} dx_i e^{-x_i^2} \Delta(x)^2 \frac{1}{N} \sum_{i=1}^{N} e^{x_i} \frac{1}{\Delta(x)} e^{-\frac{1}{4} \sum_{i=1}^{N} \partial_i^2} \Delta(x) \mathcal{O}_R(x) \tag{C.6}$$

for real eigenvalues x_i. Now, using the identity for an arbitrary function $f(x)$

$$e^{-\frac{1}{4}\partial_x^2} f(x) = e^{\frac{1}{2}x^2} f\left(\frac{x - \partial_x}{2}\right) e^{-\frac{1}{2}x^2} \tag{C.7}$$

the integral (C.6) becomes

$$\int \prod_{i=1}^{N} dx_i \, e^{-\frac{1}{2}\sum_i x_i^2} \Delta(x) \frac{1}{N} \sum_{i=1}^{N} e^{x_i} \, \mathcal{O}_R\left(\frac{x - \partial_x}{2}\right) \Delta\left(\frac{x - \partial_x}{2}\right) e^{-\frac{1}{2}\sum_i x_i^2} \tag{C.8}$$

Now, we notice that the combination $\frac{x_i - \partial_i}{\sqrt{2}} = a_i^\dagger$ is a creation operator and $e^{-\frac{1}{2}\sum_i x_i^2}$ is the wavefunction of the harmonic oscillator vacuum. The first factor and the last factor in (C.8) are both Slater determinants of the ground state wave-function of a set of N fermions moving in a harmonic oscillator potential, see eq. (A.3), up to a normalization factor

$$\mathcal{N} \Delta(x) e^{-\frac{1}{2}\sum_i x_i^2} = \mathcal{N}' \Delta\left(\frac{x - \partial_x}{2}\right) e^{-\frac{1}{2}\sum_i x_i^2} = \Psi(x_1, \ldots, x_N) \tag{C.9}$$

Using the bracket notation as in section 2.4, the integral is written as

$$\langle\Psi|\frac{1}{N}\sum_{i=1}^{N}e^{\frac{1}{\sqrt{2}}(a_i+a_i^\dagger)}\mathcal{O}_R(a^\dagger)|\Psi\rangle \tag{C.10}$$

Finally, inserting the completeness relation for coherent states, (C.10) becomes a normal matrix model in the same way as we saw in section 2.4.

This derivation of the normal matrix model shows that this relation is valid for any chiral primary $\mathcal{O}_R$. On the other hand, the Wilson loop part contains a normal ordering constant in general, hence the equivalence between complex matrix model and normal matrix model holds only for a certain representation where this normal ordering factor is common to all terms in the representation. For instance, the normal matrix model is exact for the OPE computation of the Wilson loop in the anti-symmetric representation.

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